Random walks interacting with evolving energy landscapes

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Abstract. We introduce a diffusion model for energetically inhomogeneous systems. A random walker moves on a spin-S Ising configuration, which generates the energy landscape on the lattice through the nearest-neighbors interaction. The underlying energetic environment is also made dynamic by properly coupling the walker with the spin lattice. In fact, while the walker hops across nearest-neighbor sites, it can flip the pertaining spins, realizing a diffusive dynamics for the Ising system. As a result, the walk is biased towards high energy regions, namely the boundaries between clusters. Besides, the coupling introduced involves, with respect the ordinary diffusion laws, interesting corrections depending on either the temperature and the spin magnitude. In particular, they provide a further signature of the phase-transition occurring on the magnetic lattice.

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1 Introduction

The problem of diffusion on inhomogeneous media is attracting much attention, due to its fundamental importance in nearly every field of science and engineering [1–7]. In fact, for a proper description of many diffusive systems (for example proteins "sliding" on DNA, diffusion of charge carriers in solids, flow through porous media, etc), apart from the motion of the particles, also the underlying environment must be included [8,9].

There are several ways to introduce disorder or, more generally, inhomogeneity. For example, it can be geometric, due to an irregular lattice structure, or energetic. In the latter case, the lattice sites (or bonds) are assigned different energy states and, consequently, the walker is biased towards sites corresponding to potential wells (or small energy barriers). Moreover, disorder can be deterministic or random and it can be dynamic (the environment is renewed at each jump of the walker) [1,10], or static (the environment is frozen in a particular configuration) [11,12]. Particles diffusing on such structures can also be endowed with memory effects [13,14], or be influenced by the distribution of other diffusing particles on the same structure [15].

In our model, an evolving, inhomogeneous energy landscape is introduced, by coupling the random walk with a spin-S Ising system. More precisely, we assume a spin-S arbitrary lattice and we let a random walker moving on it. The relevant energy landscape is then obtained by relating each lattice site with the pertaining nearest-neighbor

interaction according to the Ising Hamiltonian. In other words, the walker moves on a lattice where each site is occupied by a spin $\sigma_i \in [-S, S]$ and which generates the energy environment through the Ising interaction. Now, if we make the Ising ferromagnet be in contact with a heat-bath, by varying the temperature parameter the spin configuration evolves and then, also the energy landscape is modified. In particular, the temperature acts as a dispersion parameter [16], being able to control the roughness of the energetic environment. In fact, when the temperature is sufficiently low, the lattice is ferromagnetic and the energy landscape is flat, vice versa when $T \to \infty$ the energy landscape is rugged.

However, differently from the dynamic, inhomogeneous systems introduced in previous works [1,10], where the energy landscape was updated from external forces, here we assume that the random walker, while hopping across the sites of the underlying lattice, flips the relevant spins. Hence, during the diffusion of the walker on the lattice the magnetization and the energy properly vary. In fact, as we will see later, by defining a suitable spin-flip probability, the random walker is able to provide a diffusive thermal dynamics [17,18]. In particular, as a result of our assumptions, the walker is now biased towards such sites that, by flipping the relevant spin, an energy gain can be achieved.

Therefore, the problem of the RW on an inhomogeneous energy landscape is non trivially extended to the problem of their interaction: the RW affects, and is biased by, the energy landscape. In other words, there are two interplaying stochastic processes: the motion of the walker and the evolution of the spin configuration.

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Our work will be mainly numerical and the algorithm implemented is very general, being easily appliable to arbitrary lattices, made up of spins which can assume an arbitrary, finite number of states.

The aim of this work is then to characterize the random walk introduced, especially highlighting how its interaction with the magnetic lattice affects its diffusion. In particular, it would turn out to be interesting to relate the behavior of the walker with the evolution of the energy landscape, namely with the evolution of the magnetic lattice. Hence, we analyze our biased random walker (BRW) at different temperatures and then we compare results with those, already known, relevant to the ordinary, unbiased random walker (URW). Interestingly, as we will show, though their asymptotic behaviors agree [19–21], temperature dependent corrections have to be introduced. In particular, the functional laws describing the behavior of our BRW are URW-consistent, while the pertinent multiplicative factors peak at T_c. Therefore, effects due to the coupling between the walker and the magnetic system are strongest as the latter undergoes its phase transition. In other words, the diffusion of the BRW provides signatures of the phase transition occurring on the magnetic lattice. Besides, in order to understand to what extent the spin magnitude influences the walker diffusion, we take into account both spin-1/2 and spin-1 Ising systems.

Finally, notice that all the measures that are being explained are performed after the magnetic system has reached a steady state.

The layout of the paper is as follows. In Section 2 we explain how the energy landscape is generated and how our BRW can update it; we also underlines the differences with respect to the URW. In Section 3 we show how, under some conditions, such differences can vanish and then the URW is recovered. In Sections 4–7 we describe the numerical simulations performed, useful to characterize the walker behavior. We especially analyze in details the covering time, the number of returns to the origin and of distinct sites visited since they better emphasize the relationship between the walker and the magnetic lattice. Finally, Section 8 contains a summary and a discussion of results.

2 Diffusive dynamics

In this work we deal with a RW moving on, and interacting with the energy landscape generated by the following Hamiltonian applied to the magnetic configuration of a spin-S Ising system:

$$\mathcal{H} = -\frac{J}{S^2} \sum_{i,j}^N A_{ij}\,\sigma_i\sigma_j + \frac{h}{S} \sum_i^N \sigma_i. \qquad (1)$$

The spin variable σ may take the (2S+1) values $-S, -S+1, \ldots, S-1, S$ and Aij is the adjacency matrix associated to the arbitrary network where spins are placed on. Hence, the first sum only involves nearest neighbor pairs, according to the chemical distance.

Though our analysis has been performed on a toroidal squared lattice with $J = 1$, $h = 0$ in order to focus the attention on the very dynamical effects, in the remaining of this section we make assumptions on neither the structure of the lattice nor on the spin magnitude (though finite).

Hitherto we have just explained how the energy landscape is generated starting from a discrete spin configuration, while now we will describe how the coupling between the magnetic lattice and the walker works.

The random walker is assumed to be able to move on nearest-neighbor sites or stop, and it can also flip the spin pertaining to the reached site (notice that, when the spin magnitude is very large, the latter procedure can be quite complex due to a $(2S + 1)$-manifold choice). Therefore, our model displays two interplaying stochastic processes: the diffusion of the walker on the lattice and the evolution of the spin configuration. Such processes can be considered consequentially (firstly decide the site to move towards and then select the relevant spin state or vice versa) or contemporary (consider all possible combinations spin+site and choose one of them). Indeed, in any case, there exist many different ways to rule this system, ranging from completely random to completely deterministic.

The assumptions for our model have been taken in order to realize a proper diffusive dynamics for the Ising model. Such a dynamics was introduced in a previous paper [17] where the thermodynamics aspects were investigated. In particular, it was found that our diffusive thermal dynamics is actually able to drive the system towards a non canonical equilibrium state, which depends on the temperature but not on the particular initial spin configuration. As far the critical behavior, it preserves the universality class, though the critical temperature is increased:

$$T_c^{S=1/2} = 2.602(1) \qquad (2)$$

$$T_c^{S=1} = 1.955(2). \qquad (3)$$

Because of this sort of difference with respect the canonical dynamics, it is worth underlining that, in the remaining of the paper, when we refer to the critical range or temperature, it is always meant according to the diffusive dynamics.

Now, let us see in detail how the probability running our RW is defined. First of all, it contemporary takes into account the motion of the walker and the spin-flip procedure, besides, it is local since it only depends on the magnetic configuration of RW's nearest-neighbor sites. More precisely:

$$\mathcal{P}_T(\boldsymbol{s}, j | \boldsymbol{s_0}, i) = \frac{p_T(\boldsymbol{s}, j)(A_{ij} + \delta_{ij})}{\sum_{\{\boldsymbol{s'}\}} \sum_{j=0}^{z_i} p_T(\boldsymbol{s'}, j)} \qquad (4)$$

represents the probability that the walker, being on site i with coordination number z_i, jumps on a n.n. site j and realizes the magnetic configuration $\boldsymbol{s}$. The spin configuration before the jump is denoted as $\boldsymbol{s_0}$, while $\{\boldsymbol{s'}\}$ is the set of the new possible configurations. Furthermore,

$$p_T(j, \boldsymbol{s}) = \frac{1}{1 + e^{[\beta \Delta E_j(\mathbf{s})]}} \qquad (5)$$

is derived from the usual Glauber probability (see [17] for more details). Also,

$$\Delta E_j(\boldsymbol{s}) = \left(\sigma_j^i - \sigma_j^f\right) \frac{J}{S^2} \sum_{j \sim k} \sigma_k, \qquad (6)$$

is the energy variation consequent to the process, where σ_j^i and σ_j^f represent the spin-state on site j before and after the flip procedure, respectively.

You can notice that, at each step, the walker can choose among $z_i + 1$ sites to move towards (or stay on) and, contemporary, it can also choose if flip the relevant spin, being biased in order to achieve an energy gain.

Hence, all in all, there are $(z_i + 1) \times (2S + 1)$ options including that the magnetic configuration of the system, as well as the position of the walker, will possibly remain unchanged.

Of course, in a d-dimensional hypercubic lattice, the number of nearest-neighbors does not depend on the particular site and $z_i = 2d, \forall i$.

Notice that, the hopping rate between two sites is, in general, different going forward and backwards, so that the random walk is asymmetric. Consequently, as stressed in [17], this kind of dynamics does violate the detailed balance condition and the equilibrium states achieved are non-canonical.

In traditional models of diffusion on energetic landscapes, the jump rate is typically controlled by the local energy at the start point or by the energy-barrier height between start and end points [12]. Though previous equations imply that spin-flips occur on the site where the walker is moving towards, we can as well think our model in terms of energetic barriers. In fact, energy-barriers are lower for nearest-neighbor sites which let, by means of spin-flip, a higher energy gain. Furthermore, such behavior of the walker is consistent with the physical systems which have inspired the model [17,18].

It is now worth comparing our RW with the traditional unbiased random walker, usually defined according to the probability: $\mathcal{P}(i,j) = \frac{A_{ij}}{z_i}$, since, at every step the walker must move and the hopping probabilities are isotropic and do not depend on time. In this work, in order to establish a stronger analogy with our BRW, we will endow the unbiased random walker with a waiting probability so that, from site i, the possible, equivalent, choices are $z_i + 1$. In other words, we allow repetitions within the succession defining the trajectory of the walker and

$$\mathcal{P}(i,j) = \frac{A_{i,j} + \delta_{i,j}}{z_i + 1}, \qquad (7)$$

where j is a nearest neighbor of i's or, possibly, the site i itself. In the following, we will refer to this able-to-stop unbiased random walker as SURW.

It is known [22] that, for the URW, the possibility of staying on the same site is crucial in the short time regime, while in the long time behavior it has no important consequences. As we will see later, an analogous long-time effect is also experienced by our BRW.

3 BRW recovers URW

As mentioned in Section 1, the temperature parameter can tune the roughness of the energy landscape. In particular, when T is sufficiently low, the magnetic lattice is homogeneous, the energy landscape is flat and we expect to recover the URW case. On the other hand, when $T \to \infty$, a completely disordered lattice and, consequently, a rugged energy environment, is achieved. Nevertheless, since the energy variations consequent to whatever possible spin-flip would be very small compared with β, we again expect to recover the URW case. Then, in this section, we want to prove that equation (4) recovers equation (7), under the conditions $\beta \to \infty$ and $|\sum_{i=0}^{N} \sigma_i| = NS$, or $\beta \to 0$. Firstly, let us consider the former case with $\sigma_i = S, \forall i$. Suppose that the walker jumps from site i to j, with coordination numbers z_i and z_j respectively and that, consequently, σ_j is flipped in σ'. Then, equation (4) can be rewritten as

$$\mathcal{P}(i,j,\sigma_j = \sigma') = \frac{[1 + E^{z_j(S-\sigma')}]^{-1}}{\sum_{k=0}^{z_i} \sum_{l=0}^{2S} \frac{1}{1+E^{z_k l}}}, \qquad (8)$$

where $E = e^{J\beta/S}$ and i's nearest-neighbors have been numbered from 0 (the walker remains on i) to z_i. We also dropped the factor $(A_{ij} + \delta_{ij})$, because we assume j to be linked to i, or, possibly, $i = j$. Now, since $\beta \to \infty$, then $E \to \infty$ and we can write:

$$\mathcal{P}_0(i,j,\sigma_j = \sigma') = \begin{cases} \frac{1}{z_i+1} + O(E^{-\zeta}) & \text{if } \sigma' = S \\ O(E^{-z_j(S-\sigma')}) & \text{if } \sigma' \neq S, \end{cases} \qquad (9)$$

where

$$\zeta = \min_{k=0,\ldots,z_i}(z_k).$$

Conversely, when $\beta \to 0$, then $E \to 1$, and, recalling that S is finite, you can easily find that:

$$\mathcal{P}_\infty(i,j,\sigma_j = \sigma') = \frac{1}{(z_i + 1)(2S + 1)} + O(\beta J \xi), \qquad (10)$$

where

$$\xi = \max_{k=0,\ldots,z_i}(z_k).$$

The previous equation depends neither on σ', nor on the magnetic configuration of the lattice and hence, all in all, the probability of jumping from a site to another recovers equation (7).

In the following sections we analyze the behavior of the walker introduced, focusing the attention on those aspects which are mostly affected by its interaction with the magnetic lattice. Results will be further stressed by comparison with their SURW counterparts.

4 Visit lattice

In this work a task of ours is to relate the motion of the walker with the magnetic configuration of the Ising lattice
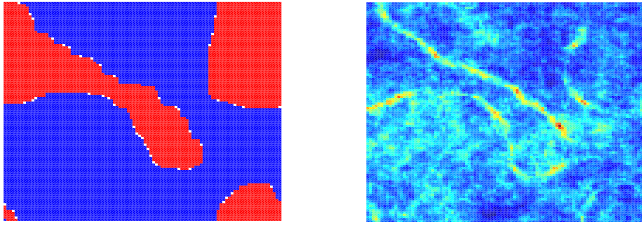
Fig. 1. (Color online) Magnetic (left panel) and Visit (right panel) lattice for a 100×100 spin-1 Ising system subject to the diffusive dynamics described in Section 2. The sample, initially paramagnetic, was suddenly cooled at low temperatures ($T < T_c$). Note that null spins (white) arrange themselves just on those sites such that their n.n. provide, all in all, a neutral magnetization; these sites usually belong to cluster boundaries. In the left panel the warmest colors are reserved to the most visited sites which are the ones corresponding to boundaries between clusters; analogous results hold for spin-1/2.

representing the energy landscape. To this aim, we introduce the visit lattice, meant as the $L \times L$ array whose elements are incremented by a unit each time the walker passes through the pertaining site. In Figure 1 such a lattice is compared with the magnetic one. From equation (4) we expect the walker to be attracted towards high energy regions which, in our model, corresponds to borders between clusters. Of course, this attraction affects the distribution of visit numbers on the lattice provided that the parameter β is not so small to make any spin-flip equally probable (see Eq. (10)). Actually, in Figure 1, the attraction felt by the walker is strong enough to generate detectable effects on the visit lattice: as expected, the most "popular sites" are just those belonging to the perimeter of cluster. As a consequence, the visit lattice mirrors the magnetic lattice: looking at the former one can derive the spin configuration and vice versa.

5 Local energy

In the previous section we showed that, according to equation (4), the BRW does not move freely, but it can be forced to stay nearby high energy regions. Hence, we expect the local energy ϵ_{loc} to be larger than the energy of the whole system ϵ. Now, we wonder if the difference between such quantities is somehow temperature dependent. Therefore, we consider the quantity $\tilde{\epsilon} = \langle \epsilon_{loc} \rangle - \langle \epsilon \rangle$, where, we recall, ϵ_{loc} represents the energy relevant to the site i occupied by the walker, namely $\epsilon_{loc} = \sigma_i \sum_{i \sim j} \sigma_j$. As shown in Figure 2, as long as the temperature is sufficiently low, the lattice appears homogeneous and $\tilde{\epsilon}$ is null. However, heating the sample, some domains develop and, since the walker verges on their borders, $\langle \epsilon_{loc} \rangle$ can increase more than $\langle \epsilon \rangle$ so that $\tilde{\epsilon}$ rises. While approaching the critical temperature, more and more clusters arise and the walker is more and more likely to be found on their boundaries, which explains the maximum in T_c. Conversely, at high temperature, when the paramagnetic phase has been reached, $\langle \epsilon \rangle$ gets to $\langle \epsilon_{loc} \rangle$.
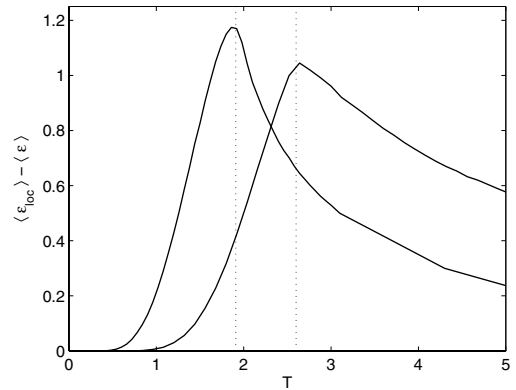
Fig. 2. $\langle \epsilon_{loc} \rangle - \langle \epsilon \rangle$ for a 240×240 spin-1/2 and spin-1 Ising systems initialized ferromagnetic and then heated. The two plots can be easily distinguished, as they peak at the relevant critical temperature (dashed line) and the former case shows a higher value of T_c.

Note also that, in Figure 2, the peak relevant to the spin-1 case is sharper, which means a stronger interaction between the walker and the magnetic lattice.

We also measure $\tilde{\epsilon}$ for an unbiased random walker allowed to rest and moving on an Ising lattice subject to a non-diffusive dynamics. Of course, in this situation, the walker is completely useless for the evolution of the system, nevertheless its behavior underlines that results obtained for the BRW are really due to its interaction with the magnetic system. In fact, we find that, for the SURW, $\tilde{\epsilon}$ remains close to zero without displaying any significant dependence on the temperature.

Therefore $\tilde{\epsilon}$ provides a signature that, as far our BRW, at the critical temperature interesting phenomena occur, not only in thermodynamics terms.

6 Covering time

The previous two sections pointed out that the hopping-flipping probability, defined in Section 1, actually biases the walker towards high energy regions and that, in the critical range, the coupling between the walker and the magnetic lattice is even more important. Hence, the phase transition also emerges from the behavior of the walker; this interesting feature will be especially taken into account in the following analysis. In particular, we now consider the covering time, namely the time (in unit step) taken by the walker to visit all N sites making up the lattice.

We recall that the lattice is squared and endowed with periodic boundary conditions, so that the walker can actually cover an infinite distance on it.

As depicted in Figure 3, for both spin-1/2 and spin-1 systems, the covering time $\mathcal{T}_N$ measured for the BRW increases with the size of the lattice and a temperature dependence is also noticeable. In particular, there is an increase in the covering time at about the critical temperature, which has been previously measured [17] revealing to be fairly larger than the canonical one (Eqs. 2 and 3).
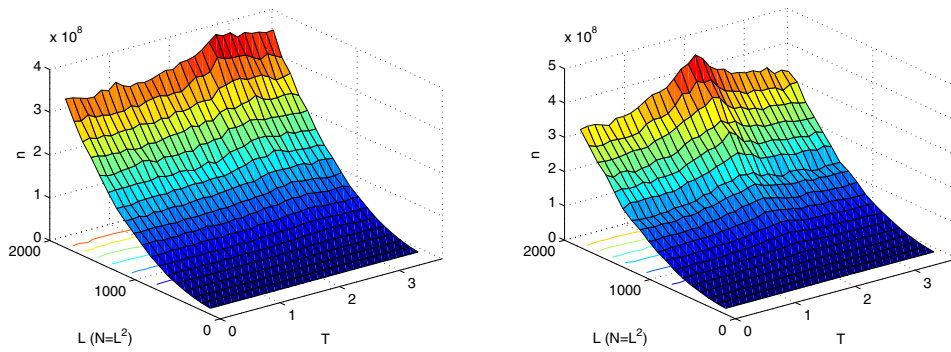
Fig. 3. (Color online) Covering time versus temperature and size of the lattice for the biased random walker moving on a spin-1/2 (left panel) and spin-1 (right panel) Ising system. In both cases the outline depends strongly on the size and the temperature affects the covering time just during the critical regime.

As far the dependence on the total number of sites N, it is consistent with the logarithmic law:

$$\mathcal{T}_N = \alpha N (\log N)^2 \tag{11}$$

found by analytical [23,24] as well as numerical [25] methods applied to an unbiased random walk. Then, the very effects due to the bias have to be tracked down in the multiplicative factor α. In fact, by fitting, according to equation (11), the data relevant to both spin systems, we evidenced that $\alpha_{\mathrm{BRW}}^{S=1/2}$ and $\alpha_{\mathrm{BRW}}^{S=1}$ depend on T and are larger than α_{URW} found in [23,25]. Notice that both the possibility of maintaining the same position and the interaction with the magnetic lattice concur in lengthening the covering time, but the role played by the latter is non trivial. In particular, it makes $\alpha_{\mathrm{BRW}}^{S=1/2}$ and $\alpha_{\mathrm{BRW}}^{S=1}$ exhibit a maximum at about T_c, namely, in the critical region, it takes more time for the BRW to cover the lattice. On the other hand, the SURW displays a covering time still consistent with equation (11) but, of course, independent on T. Note that, as shown in Figure 4, within the error ($<2\%$),

$$\alpha_{\mathrm{SURW}} \leq \alpha_{\mathrm{BRW}}^{S=1/2} \leq \alpha_{\mathrm{BRW}}^{S=1}. \tag{12}$$

As expected, the BRW, with respect the SURW, is slowed down since it may be "trapped" nearby high energy regions constituted by sites where several, energetically favorable, spin-flips are possible. On the other hand, the quantities in equation (12) are all comparable at low temperatures. In fact, during the ferromagnetic phase, when $T \ll T_c$, the lattice appears homogeneous and the bias has no effect; an analogous phenomenon is expected at very high temperatures (in Sect. 3 we proved that, indeed, in these cases the BRW recovers the SURW). Besides, since in the spin-1 case the walker has to manage a greater number of possibilities, the slowing down effect is even higher and the relevant peak in Figure 4 is more marked.

Note that, with a non-diffusive dynamics, the time required to scan each lattice site at least once can be much smaller. For example, by adopting the type-writer sequence, $\mathcal{T}_N$ is reduced by a factor $4\alpha_{\mathrm{BRW}}(\log L)^2$. For this reason, our diffusive dynamics may be though as "slow".
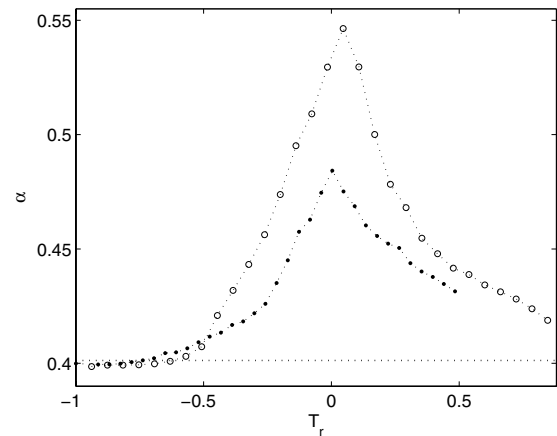

Fig. 4. Multiplicative constant α versus reduced temperature $T_r = \frac{T - T_c}{T_c}$ for the BRW applied to the spin-1/2 ($\bullet$) and spin-1 ($\circ$) Ising systems. Both functions peaks at zero, while α_{URW} (dashed line) is temperature independent. Note that (within the error) $\alpha_{\mathrm{SURW}} \leq \alpha_{\mathrm{BRW}}^{S=1/2} \leq \alpha_{\mathrm{BRW}}^{S=1}$.

7 Returns to the origin and distinct sites visited

In this section, we want to deal with other two characteristic quantities concerning the BRW: the number of returns to the origin R_n and of distinct sites visited D_n, after an n-step walk. Actually, we ought to distinguish between two regimes:

1. if n is a step number satisfying $n \ll N$, since the walker will not have sampled a substantial number of sites of the lattice, the lattice will appear to be infinite;
2. in the long time ($N \ll n$) the walker will appreciate the toroidal effect due to the periodic boundary conditions.

Therefore, according to the walk-length, our results will be compared with those analytically known and relevant to the URW on an infinite or periodic 2-dimensional lattice, respectively. As we will see, in both ranges, the exponents found for R_n and D_n agree with known results, while, as far the multiplicative factors, one has to introduce a dependence on T.
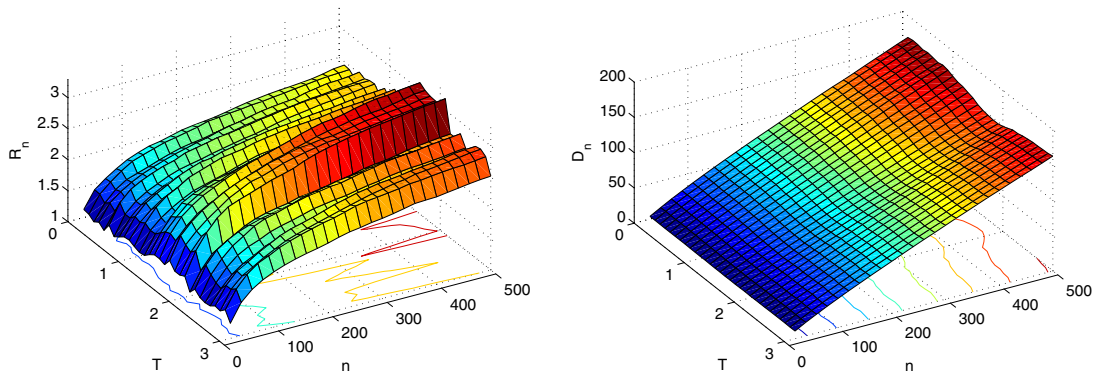
Fig. 5. (Color online) Average number of returns to the origin (left panel) and of distinct sites visited (right panel) versus temperature and number of steps made by the BRW on a 240×240 spin-1 Ising lattice. Analogous results were found for the spin-1/2 system, though the critical phenomena are less emphasized. Note that, as $n \ll N$, the walker can not realize the finiteness of the lattice.
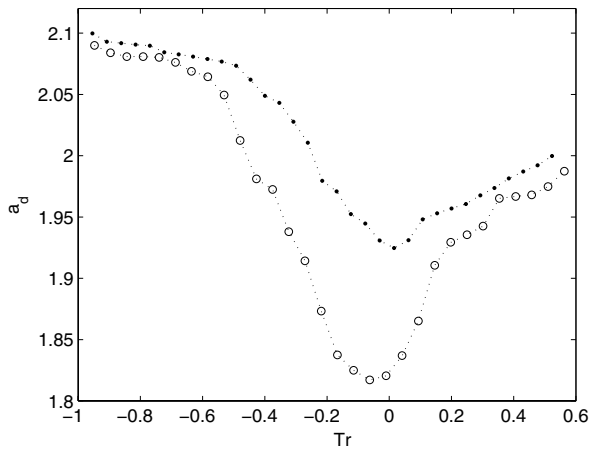
Fig. 6. Fit parameter a_D versus reduced temperature relevant to the BRW on a 240×240 spin-1/2 and spin-1 Ising lattice. Both functions exhibit a minimum at about the relevant critical temperature ($T_c \approx 2.6$ and $T_c \approx 1.96$ respectively) which is deeper in the latter case. Conversely, at low temperatures, plots overlap.

Let us firstly consider the short time case. Known results [19,20] about the average number of returns to the origin R_n and the number of distinct sites visited D_n, after an n-step walk for an URW, state that:

$$R_n \sim a_R \log n, \qquad (13)$$

$$D_n \sim \frac{a_D n}{\log n}, \qquad (14)$$

where a_R and a_D are constant. By fitting and comparing our outcomes relevant to spin-1/2 and spin-1 systems (Fig. 5), we find that equations (13) and (14) formally still hold, but a_R and a_D are functions of the temperature. More precisely, in T_c they show a maximum and a minimum, respectively; such effect is more important when $S = 1$ (Fig. 6).

When the walk-length is large enough, for the walker, to experience the toroidal effect, we find that the number

of returns R_n recovers the URW case, being

$$R_n \sim \frac{n}{N}, \qquad (15)$$

while the number of distinct sites visited is consistent with

$$D_n \sim N \left(1 - e^{-nA_D}\right), \qquad (16)$$

relevant to the URW on a periodic lattice [20], provided that A_D depends on T (Fig. 8)

However, by further increasing n, the bias effect vanishes also for D_n which, finally, equals N.

Therefore, our results are asymptotically independent on the temperature and on the spin magnitude, which means that the walker looses memory of its bias.

It should be underlined that, in both regimes, the extreme points recorded at T_c for R_n, as well as D_n, are consistent with what previously found about $\mathcal{T}_N$ and $\tilde{\epsilon}$.

Analogous results are expected also for the probability of return to the origin $P_{0,n}$. In fact, we anticipate that the BRW still recovers the unbiased exponent, with a multiplicative factor maximum at T_c.

Notice that recovering the conventional diffusive regime from a significantly altered model (at least in a long time limit) is consistent with several previous works [1,26]. The fact that our model yields diffusive behavior should be related to the absence of strong memory effects which, indeed, could determine an anomalous diffusion [13,14].

8 Conclusions

By the analysis performed so far, we are able to characterize the BRW introduced and also to relate its behavior with the evolution of the underlying energetic environment.

Our measures of local energy show that, according to the algorithm introduced in Section 2, the walker aims to move towards high energy regions, where favorable spin-flips can occur. These regions correspond to boundaries between clusters, whose concentration depends on
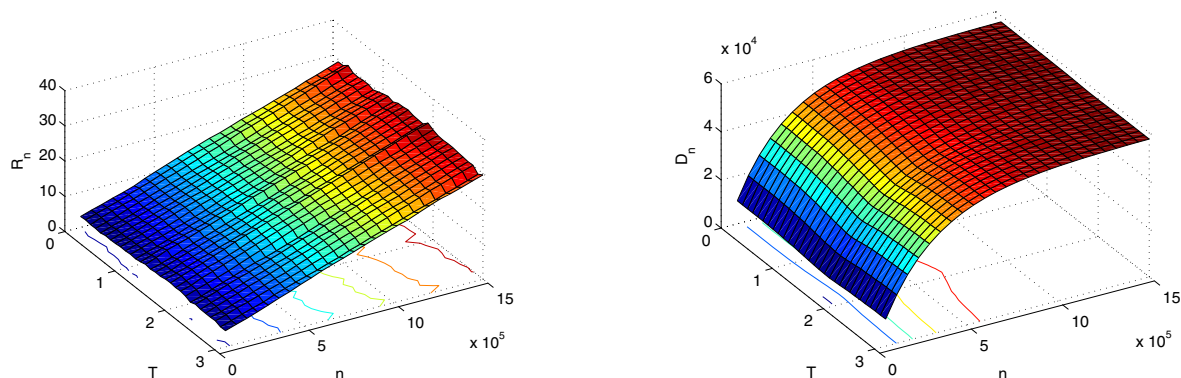
Fig. 7. (Color online) Average number of returns to the origin (left panel) and of distinct sites visited (right panel) versus temperature and number of steps n made by the BRW on a 240×240 spin-1 Ising lattice. In this case $N \ll n$ and the walker can experience the periodic boundary conditions the lattice is endowed with. Note that the temperature dependence is now scarcely detectable. Similar results were found for the spin-1/2 system.

the temperature, being largest around T_c. Thus, there exist a sort of temperature sensitive traps, the walker is attracted to. This attraction reflects on a non-homogeneous visit numbers distribution and also causes a slowing down well evinced by measures of covering time, numbers of returns to the origin and of distinct sites visited. In general, these quantities asymptotically agree with results, analytically known, relevant to the URW. In particular, when the walk-length is not so large to visit all lattice sites, their functional forms are URW-consistent, but temperature dependent multiplicative factors have to be introduced. Their particular dependence on T has two main consequences: first, the slowing down is not only due to a non-null waiting probability, but it is mainly a consequence of the bias introduced; second, the interaction between the walker and the magnetic lattice is stronger in the critical region. More precisely, in that temperature range, the length of borders between clusters is large so that there are lots of high-energy sites and, contemporary, the temperature is still not too large to have a significant (with respect β) energy gain by spin-flips. Hence, multiplicative factors just peak at the critical temperature.

What has been said hitherto holds for either spin-1/2 and spin-1 systems. In fact, our analysis have been contemporary performed on both, in order to evidence how the spin magnitude affects the walker diffusion on the lattice. Then, our results show that peaks get sharper and higher when the number of spin states is larger. Actually, in the latter case, the walker has to manage with more opportunities and, consequently, it is further slowed down.

In summary, in the limit $T \to 0$ ($T \to \infty$) the underlying energetic landscape becomes homogeneous and the BRW recovers the case of an ordinary unbiased random walker endowed with a non-null waiting probability. On the other hand, when $T = T_c$ we record the most important effects. In fact, T_c is an extremal point for the multiplicative factors which correct the ordinary laws. Hence, the bias introduced determines stronger effects as the critical temperature is approached, though not affecting the diffusive regime. Therefore the BRW behavior gives a further evidence of the phase transition.
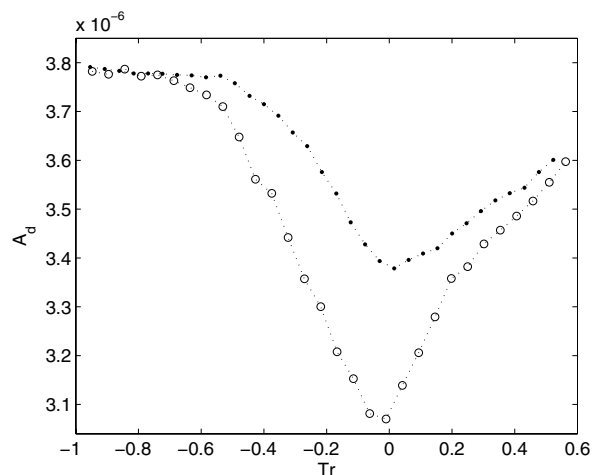
Fig. 8. Fit parameter A_D versus reduced temperature pertaining to the BRW on a 240×240 spin-1/2 and spin-1 Ising lattice. Note that the minimum occurs at about the critical temperature ($T_c \approx 2.6$ and $T_c \approx 1.96$ respectively) and it is deeper in the latter case. Conversely, at low temperatures, plots are overlapped.

Since such corrections are more important when the spin magnitude is larger, we argue that an investigation in the continuum limit for S would turn out to be useful in order to clear the nature of the above mentioned extreme points.

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